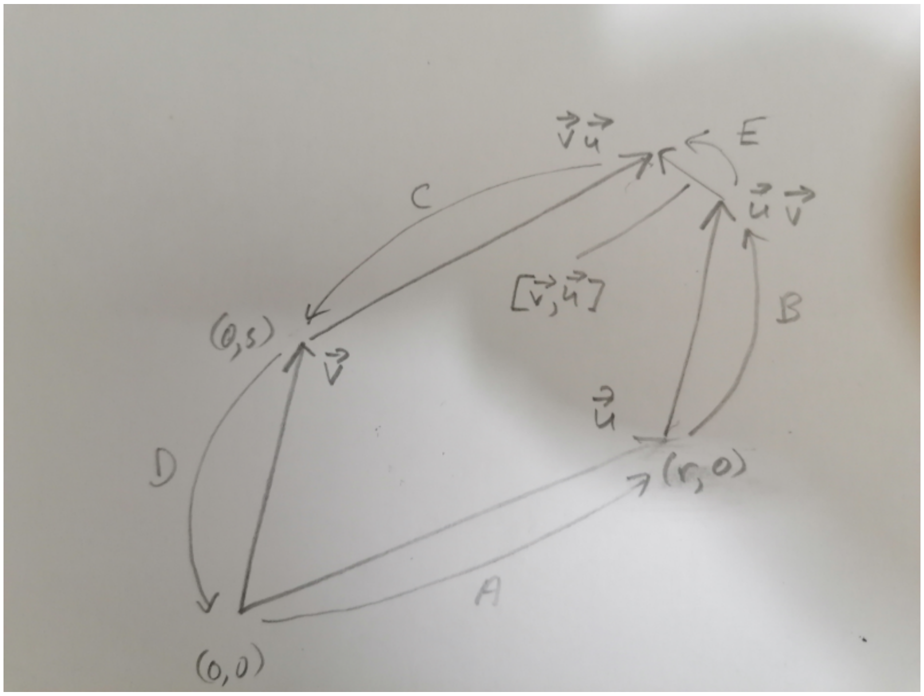

Figure 1: Open Parallellogram

1 The Derivation of the Formula for the Riemann Curvature Tensor in the Presence of Torsion

The Riemann curvature tensor $R(\vec{u}, \vec{v})$, in a space *without* torsion, is derived by applying parallel transport to a vector $\vec{w}$, anticlockwise around a parallelogram, and comparing the result to the original vector. When there is torsion, the parallelogram doesn't close and there is an extra edge $[\vec{v}, \vec{u}]$ along which there has to be additional parallel transport. The value of $R(\vec{u}, \vec{v})$ does not depend on the starting point.

Various sources give the value of $R(\vec{u}, \vec{v})$ to be

$$R(\vec{u}, \vec{v}) = [\nabla_{\vec{u}}, \nabla_{\vec{v}}] - \nabla_{[\vec{u}, \vec{v}]} \tag{1}$$

However, none of the sources I have seen provides a rigorous derivation of this.

The derivation in what follows is based on the one given in eigenchris's Youtube lecture course, and in particular on https://www.youtube.com/watch?v=-Il2FrmJtcQ (Tensor Calculus 22).

In a space with torsion, the Riemann curvature tensor is defined as (see

Figure 1)

$$\begin{aligned} R(\vec{u},\vec{v})\vec{w} &= \\ \lim_{r,s\to 0}\frac{\vec{w}-EBADC\vec{w}}{rs} &= \\ \lim_{r,s\to 0}\frac{E[E^{-1}\vec{w}-BADC\vec{w}]}{rs} &= \\ \lim_{r,s\to 0}E[\frac{E^{-1}\vec{w}-\vec{w}}{rs}-\frac{BADC\vec{w}-\vec{w}}{rs}] \end{aligned} \tag{2}$$

The first term is the correction for torsion. The second term is the torsion-free term, which we can expand as follows:

$$\begin{aligned} \frac{BADC\vec{w}-\vec{w}}{rs} &= \\ BA\frac{DC\vec{w}-A^{-1}B^{-1}\vec{w}}{rs} &= \\ BA\frac{(DC\vec{w}-D\vec{w}+D\vec{w}-\vec{w})-(A^{-1}B^{-1}\vec{w}-A^{-1}\vec{w}+A^{-1}\vec{w}-\vec{w})}{rs} &= \\ BA[(\frac{D}{s}\frac{C\vec{w}-\vec{w}}{r}-\frac{1}{r}\frac{D\vec{w}-\vec{w}}{s})-(\frac{A^{-1}}{r}\frac{B^{-1}\vec{w}-\vec{w}}{s}+\frac{1}{s}\frac{A^{-1}\vec{w}-\vec{w}}{r})] \end{aligned} \tag{3}$$

The four difference terms can be converted into covariant derivatives:

$$\frac{A^{-1}\vec{w}-\vec{w}}{r} = \nabla_{-\vec{u}}\vec{w} = -\nabla_{\vec{u}}\vec{w} \tag{4}$$

$$\frac{B^{-1}\vec{w}-\vec{w}}{s} = \nabla_{-\vec{v}}\vec{w} = -\nabla_{\vec{v}}\vec{w} \tag{5}$$

$$\frac{C\vec{w}-\vec{w}}{r} = \nabla_{-\vec{u}}\vec{w} = -\nabla_{\vec{u}}\vec{w} \tag{6}$$

$$\frac{D\vec{w}-\vec{w}}{s} = \nabla_{-\vec{v}}\vec{w} = -\nabla_{\vec{v}}\vec{w} \tag{7}$$

Substituting 4, 5, 6, and 7, into 3, we get

$$\begin{aligned} \frac{BADC\vec{w}-\vec{w}}{rs} &= \\ BA[-\frac{D}{s}\nabla_{\vec{u}}\vec{w}-\frac{1}{r}\nabla_{\vec{v}}\vec{w}+\frac{A^{-1}}{r}\nabla_{\vec{v}}\vec{w}+\frac{1}{s}\nabla_{\vec{u}}\vec{w}] &= \\ BA[-\frac{D\nabla_{\vec{u}}\vec{w}-\nabla_{\vec{u}}\vec{w}}{s}+\frac{A^{-1}\nabla_{\vec{v}}\vec{w}-\nabla_{\vec{v}}\vec{w}}{r}] \end{aligned} \tag{8}$$

and the remaining two difference terms can also be converted into covariant derivatives:

$$\frac{D\nabla_{\vec{u}}\vec{w}-\nabla_{\vec{u}}\vec{w}}{s} = -\nabla_{\vec{v}}\nabla_{\vec{u}}\vec{w} \tag{9}$$

$$\frac{A^{-1}\nabla_{\vec{v}}\vec{w} - \nabla_{\vec{v}}\vec{w}}{r} = -\nabla_{\vec{u}}\nabla_{\vec{v}}\vec{w} \tag{10}$$

Substituting 9 and 10 into 8, the torsion-free term becomes

$$\begin{aligned} \frac{BADC\vec{w} - \vec{w}}{rs} &= \\ BA(\nabla_{\vec{v}}\nabla_{\vec{u}}\vec{w} - \nabla_{\vec{u}}\nabla_{\vec{v}}\vec{w}) &= \\ -BA(\nabla_{\vec{u}}\nabla_{\vec{v}}\vec{w} - \nabla_{\vec{v}}\nabla_{\vec{u}}\vec{w}) \end{aligned} \tag{11}$$

The correction for torsion is (cf. 4, 5, 6, and 7, and noting that unlike A, B, C, or D, E is a difference of products of $\vec{u}$ and $\vec{v}$, and so requires the product of r and s to normalize)

$$\frac{E^{-1}\vec{w} - \vec{w}}{rs} = -\nabla_{[\vec{v}\vec{u}-\vec{u}\vec{v}]}\vec{w} = \nabla_{[\vec{u}\vec{v}-\vec{v}\vec{u}]}\vec{w} = \nabla_{[\vec{u},\vec{v}]}\vec{w} \tag{12}$$

Substituting 11 and 12 into 2, yields the definition of the Riemann curvature tensor. Note that, as r and s tend to zero, the changes A, B, C, D, and E tend to the identity operator, so no longer appear in the final result.

$$\begin{aligned} R(\vec{u},\vec{v})\vec{w} &= \\ \lim_{r,s\to 0} E[\frac{E^{-1}\vec{w} - \vec{w}}{rs} - \frac{BADC\vec{w} - \vec{w}}{rs}] &= \\ \lim_{r,s\to 0} E[\nabla_{[\vec{u},\vec{v}]}\vec{w} + BA(\nabla_{\vec{u}}\nabla_{\vec{v}}\vec{w} - \nabla_{\vec{v}}\nabla_{\vec{u}}\vec{w})] &= \\ \nabla_{[\vec{u},\vec{v}]}\vec{w} + \nabla_{\vec{u}}\nabla_{\vec{v}}\vec{w} - \nabla_{\vec{v}}\nabla_{\vec{u}}\vec{w} &= \\ [\nabla_{\vec{u}}, \nabla_{\vec{v}}]\vec{w} + \nabla_{[\vec{u},\vec{v}]}\vec{w} \end{aligned} \tag{13}$$

or

$$R(\vec{u},\vec{v}) = [\nabla_{\vec{u}}, \nabla_{\vec{v}}] + \nabla_{[\vec{u},\vec{v}]} \tag{14}$$

The first term is unchanged, but the torsion correction has the opposite sign. Why?

Note that beginning the parallel transport at $(0, 0)$ leaves us no better off:

$$
\begin{aligned}
R(\vec{u},\vec{v}) &= \\
\lim_{r,s\to 0}\frac{\vec{w}-DCDEBA\vec{w}}{rs} &= \\
\lim_{r,s\to 0}\frac{\vec{w}-DCBA\vec{w}}{rs}+\lim_{r,s\to 0}\frac{DCBA\vec{w}-DCDEBA\vec{w}}{rs} &= \\
\nabla_{\vec{u}}\nabla_{\vec{v}}\vec{w}-\nabla_{\vec{v}}\nabla_{\vec{u}}\vec{w}+\lim_{r,s\to 0}DC\frac{BA\vec{w}-EBA\vec{w}}{rs} &= \\
[\nabla_{\vec{u}},\nabla_{\vec{v}}]\vec{w}+\lim_{r,s\to 0}DC\frac{\vec{w}'-E\vec{w}'}{rs} &= \\
[\nabla_{\vec{u}},\nabla_{\vec{v}}]\vec{w}-\lim_{r,s\to 0}DC\frac{E\vec{w}'-\vec{w}'}{rs} &= \\
[\nabla_{\vec{u}},\nabla_{\vec{v}}]\vec{w}-\lim_{r,s\to 0}DC\frac{E\vec{w}'-\vec{w}'}{rs} &= \\
[\nabla_{\vec{u}},\nabla_{\vec{v}}]\vec{w}-\lim_{r,s\to 0}DC\nabla_{\vec{v}\vec{u}-\vec{u}\vec{v}}\vec{w}' &= \\
[\nabla_{\vec{u}},\nabla_{\vec{v}}]\vec{w}-\lim_{r,s\to 0}DC\nabla_{[\vec{v},\vec{u}]}\vec{w}' &= \\
[\nabla_{\vec{u}},\nabla_{\vec{v}}]\vec{w}+\lim_{r,s\to 0}DC\nabla_{[\vec{u},\vec{v}]}\vec{w}' &= \\
[\nabla_{\vec{u}},\nabla_{\vec{v}}]\vec{w}+\nabla_{[\vec{u},\vec{v}]}\vec{w}' &
\end{aligned}
\tag{15}
$$

after substituting $\vec{w}'$ for $BA\vec{w}$. But now, not only do we still have the $+$ sign on the $\nabla_{[\vec{u},\vec{v}]}$ term, we have $\vec{w}'$ when we should have $\vec{w}$.